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A method to determine the electroweak mixing angle from Z decays to tau leptons using optimal observables

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Abstract: We propose a new method to measure the electroweak mixing angle in Z decays to tau leptons. For various tau decay channels optimal observables are constructed which yield the largest possible signal-to-noise ratio. These optimal observables are derived from the full spin-density-matrix of the process including correlations between tau spins and an approximation to the tau flight direction.

Keywords: Standard Model, Weak Decays, LEP HERA and SLC Physics.

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Contents

1 Introduction

1.1 The electroweak mixing angle

The electroweak mixing (Weinberg) angle θ_w is an essential parameter of the Standard Model (SM). It connects the gauge coupling constant g_1 and g_2 of the electroweak $\mathrm{U(1)_Y}\times\mathrm{SU(2)_L}$ theory through the tree level relation

$$\tan\theta_\mathrm{w} = \frac{g_1}{g_2} \ . \qquad (1.1)$$

In the framework of the SM with one Higgs doublet Eq. (1.1) can be rewritten using the masses of the gauge bosons Z and $W^{\pm}$ as

$$\sin^2\theta_{\mathrm{w}} = 1 - \frac{m_{\mathrm{W}}^2}{\rho\, m_{\mathrm{Z}}^2}\,, \tag{1.2}$$

where $\rho = 1$ corresponds to the on-shell definition of the weak mixing angle. A commonly used ansatz [1] to incorporate electroweak corrections into the calculation of asymmetries is the introduction of the effective vector and axial vector coupling constants

$$\begin{aligned} \bar{V}^f &= \sqrt{\rho^f}\,(T_3^f - 2Q^f\kappa^f\sin^2\theta_{\mathrm{w}})\,, \\ \bar{A}^f &= \sqrt{\rho^f}\,T_3^f\,, \end{aligned} \tag{1.3}$$

where the electroweak corrections are absorbed into the coefficients ρ^f and κ^f which depend on the fermion f and on the renormalization scheme. Here T_3^f is the third component of the weak isospin, and Q^f is the electric charge of the fermion. By defining an effective mixing angle

$$\sin^2\theta_{\mathrm{eff}}^f = \kappa^f\sin^2\theta_{\mathrm{w}} \tag{1.4}$$

$\bar{V}^f$ and $\bar{A}^f$ are expressed by ρ^f times their tree level values. This has the advantage that tree level calculations merely have to be modified by replacing the tree level coupling constants by the effective coupling constants in order to account for electroweak corrections.

1.2 Experimental results

One of the major achievements of LEP is the measurement of the Weinberg angle with very high precision. The current value of the combined LEP measurements [2]

$$\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}} = 0.23200 \pm 0.00027 \tag{1.5}$$

obtained from combining many different asymmetry measurements (see Tab. 1), has improved the precision by two orders of magnitude compared to the situation before LEP and SLC started.

leptonic channels	$\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$
$A_{\mathrm{F}B}^{0,l}$	0.23085 ± 0.00056
A_τ	0.23240 ± 0.00085
A_e	0.23264 ± 0.00096
hadronic channels	
$A_{\mathrm{F}B}^{0,b}$	0.23246 ± 0.00041
$A_{\mathrm{F}B}^{0,c}$	0.23155 ± 0.00112
$\langle Q_{\mathrm{F}B}\rangle$	0.2320 ± 0.0010

Table 1: Values of $\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$ extracted from asymmetries measured at LEP [2].

Complementary to the asymmetries investigated at LEP, the SLD collaboration measures the left-right asymmetry A_{LR}, exploiting SLC's polarized electron beam. The value obtained by SLD is $\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}} = 0.23060 \pm 0.00050$ [3], which is the most precise single measurement and differs by more than two standard deviations from the global fit to all existing measurements [2]. The value of the electroweak mixing angle has a strong impact - obtained from radiative corrections - on the limit on the Higgs mass. A fit of

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SM parameters [4] which uses all relevant measurements including A_{LR} [3] gives an upper limit $m_H < 364$ GeV (95% c.l.), whereas a similar fit leads to $m_H < 622$ GeV (95% c.l.) when A_{LR} is excluded.

1.3 Methods to determine $\sin^2\theta_w$

In this paper we discuss so called 'optimal observables' for the measurement of $\sin^2\theta_w$ in the process $e^+e^- \to Z \to \tau^+\tau^-$ with subsequent τ decays. The method combines the whole experimentally available information of this reaction to a single one-dimensional observable. In this way it is complementary to the well known techniques which exploit only parts of the kinematical information such as the forward-backward asymmetry A^τ_{FB}, the average τ polarization P_τ, and the forward-backward polarization asymmetry A^{pol}_{FB}. Various methods to measure these quantities are described in [5, 6, 7, 8, 9]. From these measurements, which are considered as statistically independent, one derives a value of $\sin^2\theta_w$ within the framework of the SM. Optimal observables for the measurement of P_τ from single-particle energy spectra have been investigated in [10] and are used by the four LEP experiments [11, 12, 13, 14]. The purpose of the observables proposed in this article is the direct measurement of $\sin^2\theta_w$ using the whole experimentally available information in the statistically optimal way. We derive these observables from the spin-density-matrix of the intermediate $\tau^+\tau^-$ state which contains more information on $\sin^2\theta_w$ than the averaged τ polarization as investigated in previous analyses. In order to obtain the optimal observables we use an approximation to the τ flight direction.
Recently the OPAL collaboration [15] has presented results on $\sin^2\theta^{lept}_{eff}$ applying the method proposed in this paper. Optimal observables have also been used to test CP violation in τ production at LEP [16, 17, 18].
In section 2 we outline the method of optimal observables, give explicit formulas for the process considered here, and discuss the sensitivity with which $\sin^2\theta_w$ can be extracted from the data. Numerical results are summarized in section 3.

2 Optimal observables for $\sin^2\theta_w$

2.1 Concept of optimal observables

In the following we present a general ansatz to extract $\sin^2\theta_w$ with lowest possible statistical uncertainty, using the full information of the matrix element of the process

$$e^+(\mathbf{p}) + e^-(-\mathbf{p}) \to Z \to \tau^+(\mathbf{k}) + \tau^-(-\mathbf{k}) \tag{2.1}$$

with subsequent decays $\tau \to e\nu_e\nu_\tau, \mu\nu_\mu\nu_\tau, \pi\nu_\tau, \rho\nu_\tau$. These channels cover about 72% of all possible τ decays.

The dependence of the cross section (2.1) on $\sin^2\theta_w$ is induced by the interference of the neutral vector and axial vector couplings given in (1.3). Generally, in order to measure this effect one has to pick out the terms of the differential cross section which are proportional to the product $\bar{V}^f\bar{A}^f$. For this purpose we derive the observable $\mathcal{O}$

whose expectation value is proportional to $\bar{V}^f\bar{A}^f$ and which yields the largest possible signal-to-noise ratio defined by

$$R_{\mathcal{O}} = \frac{|\langle\mathcal{O}\rangle|}{\sqrt{\langle\mathcal{O}^2\rangle - \langle\mathcal{O}\rangle^2}} \,. \tag{2.2}$$

By the variation of $R_{\mathcal{O}}$ with respect to $\mathcal{O}$ and assuming parity invariant phase space cuts one finds that the maximum of $R_{\mathcal{O}}$ is provided by observables of the form

$$\mathcal{O} \propto \frac{\mathrm{d}\sigma_{\mathrm{P}_{odd}}}{\mathrm{d}\sigma_{\mathrm{P}_{even}}} \,. \tag{2.3}$$

Here $\mathrm{d}\sigma_{\mathrm{P}_{odd}}$ are the terms in the differential cross section which are parity-odd and hence proportional to $\bar{V}^f\bar{A}^f$. The remaining terms of the differential cross section are denoted by $\mathrm{d}\sigma_{\mathrm{P}_{even}}$. Observables of the form (2.3) give the lowest possible statistical uncertainty and contain all desired information in a one-dimensional function. This avoids multi-dimensional phase space fits, which are inevitable if the information is not carried in one single kinematical variable.

The spin-density-matrix of reaction (2.1) has been calculated in [19] for unpolarized e^+ and e^- beams (see Appendix A). With the assumptions of lepton universality ($V = V^{\mathrm{e}} = V^{\tau}$ and $A = A^{\mathrm{e}} = A^{\tau}$), Z exchange only, and by expanding the spin-density-matrix up to first order in the vector coupling constant, we construct the one-dimensional optimal observable for the measurement of $\sin^2\theta_{\mathrm{w}}$ by dividing the parity violating terms $\mathrm{M}_{\mathrm{P}_{odd}}$ ($\propto AV$) of the spin-density-matrix by the parity conserving terms $\mathrm{M}_{\mathrm{P}_{even}}$ ($\propto A^2$). In principle, also terms proportional to V^2 carry information about $\sin^2\theta_{\mathrm{w}}$, but due to the small absolute value of the vector coupling V compared to the axial coupling A, they can be neglected without loss of sensitivity. The optimal observable $\mathcal{O}_{opt}$ is then given by

$$\mathcal{O}_{opt} = \frac{\mathrm{M}_{\mathrm{P}_{odd}}}{\mathrm{M}_{\mathrm{P}_{even}}} \tag{2.4}$$

$$\begin{aligned}
\mathrm{M}_{\mathrm{P}_{odd}} &= \hat{\mathbf{k}}\cdot(\mathbf{S}^+ + \mathbf{S}^-)(k_0 + (k_0 - m_\tau)(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2) + \\
&\quad m_\tau(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}\cdot(\mathbf{S}^+ + \mathbf{S}^-) + 2|k|(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\hat{\mathbf{k}}\cdot(\mathbf{S}^+ + \mathbf{S}^-) && (2.5)\\
\mathrm{M}_{\mathrm{P}_{even}} &= 1 + (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 + \mathbf{S}^+\cdot\mathbf{S}^-(1 - (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2) - \\
&\quad 2\hat{\mathbf{p}}\cdot\mathbf{S}^+\hat{\mathbf{p}}\cdot\mathbf{S}^- + 2\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}(\hat{\mathbf{k}}\cdot\mathbf{S}^+\hat{\mathbf{p}}\cdot\mathbf{S}^- + \hat{\mathbf{k}}\cdot\mathbf{S}^-\hat{\mathbf{p}}\cdot\mathbf{S}^+) && (2.6)
\end{aligned}$$

which has already been shown in [20]. Hats denote unit momenta, $\mathbf{k}$ and $\mathbf{p}$ are the τ^+ and e^+ momenta, respectively, and $\mathbf{S}^+, \mathbf{S}^-$ are the spin vectors of the τ^+ and τ^- (see Appendix C). These spin vectors, first calculated in [20], describe the expectation value of the τ spin in the τ rest system. Since the $\mathbf{S}^\pm$ depend on the τ decay mode, we get for each $\tau^+\tau^-$ decay channel a different optimal observable.

2.2 Sensitivity on $\sin^2\theta_{\mathrm{w}}$

The mean value of the optimal observable $\mathcal{O}_{opt}$ is sensitive to parity violation, which vanishes for $\sin^2\theta_{\mathrm{w}} = 0.25$. Therefore in a small region around this value using the mean of the optimal observable $\langle \mathcal{O}_{opt} \rangle$ is almost equivalent regarding its sensitivity to a maximum likelihood fit (see also discussion in [10]). The expansion of the spin-density-matrix up to the first order in the vector coupling constant exhibits the following linear dependence (shown in Fig. 1),

$$\langle \mathcal{O}_{opt}(\theta_\mathrm{w}) \rangle = a + b \sin^2 \theta_\mathrm{w} \tag{2.7}$$

with real constants a and b which we calculate numerically. We define the statistical sensitivity S on $\sin^2 \theta_\mathrm{w}$ as

$$S = \sqrt{\frac{1}{N\sigma^2}} \tag{2.8}$$

where N is the number of events and σ the statistical error on $\sin^2 \theta_\mathrm{w}$. Using the linear relation (2.7) and taking the root mean square of the optimal observable distribution as statistical error on $\langle \mathcal{O}_\mathrm{opt} \rangle$, the sensitivity S with respect to $\sin^2 \theta_\mathrm{w}$ is given by

$$S = \frac{b}{\sqrt{\langle {\mathcal{O}_\mathrm{opt}}^2 \rangle - \langle \mathcal{O}_\mathrm{opt} \rangle^2}} \,. \tag{2.9}$$

where errors on the parameters a and b have been neglected, because they can be arbitrarily reduced by increasing the Monte Carlo statistics.

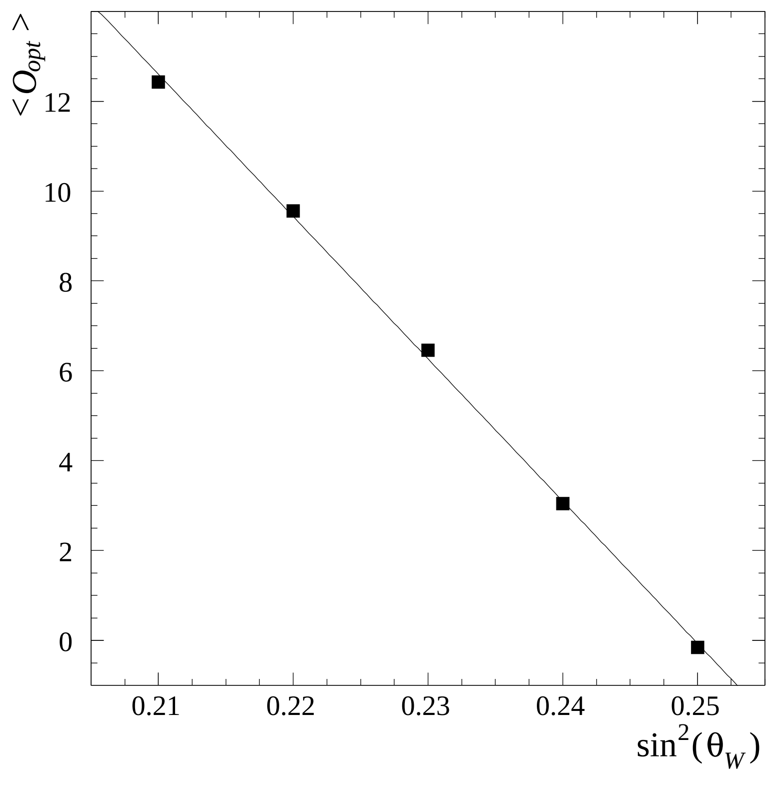

Figure 1: Linear fit to $\langle \mathcal{O}_\mathrm{opt} \rangle$ obtained from independently produced $\pi - \pi$ channel Monte Carlo events for five values of $\sin^2 \theta_\mathrm{w}$ without applying radiative corrections.

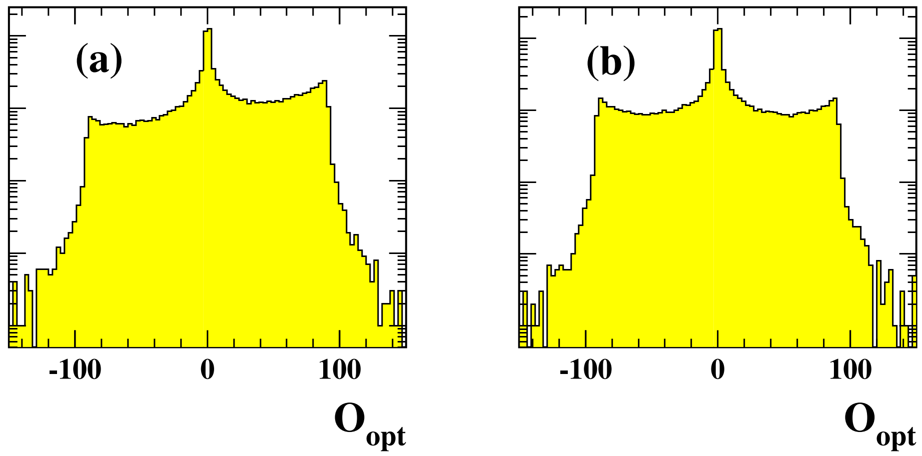

Figure 2: Distribution of $\mathcal{O}_{opt}$ for (a) $\sin^2\theta_\mathrm{w} = 0.21$ and (b) $\sin^2\theta_\mathrm{w} = 0.25$ in the $\pi - \pi$ channel for 100 000 Monte Carlo events without radiative corrections.

2.3 Distributions of optimal observables

Figure 2 shows distributions of the optimal observable obtained from events generated with the KORALZ4.0/TAUOLA2.4 Monte Carlo program [21, 22], without applying radiative corrections. The value of the optimal observable is limited due to phase space and kinematic constraints and the spectrum is symmetric for $\sin^2\theta_\mathrm{w} = 0.25$, where the vector coupling vanishes. These observations are strictly only true for tree level spectra and must be weakened when radiative corrections and detector effects are applied.

3 Numerical results

3.1 Sensitivities

We have calculated the sensitivity (2.9) for several τ decay channels using a Monte Carlo event generator [21, 22] neglecting radiative corrections (see Tab. 2). Taking initial and final state radiation and the $\gamma - \mathrm{Z}$ interference into account reduces the sensitivity in the order of a few per cent (e.g. 4% in the $\pi - \pi$ channel).
If the τ flight direction were known exactly, the sensitivities would be equal for all hadronic τ decays (cf. [23]). This has been verified by Monte Carlo studies for the $\pi - \pi$ and $\rho - \rho$ channel using the method proposed in this paper. However, in reality the τ flight direction cannot be measured directly but must be approximately deduced from the final state particles. This decreases the sensitivity, dependent on the decay mode. In this paper we use the formulas quoted in Appendix B to reconstruct the τ flight direction from the momenta of the τ decay products. Therefore the sensitivities shown in Tab. 2 have been calculated using the momenta of the final state particles and averaging over the ambiguity as explained in Appendix B.

The method of optimal observables has been used to measure the τ polarization from one-prong energy spectra [10]. To compare this with our results one has to extrapolate

channel	sensitivity to $\sin^2\theta_{\rm w}$
$\pi - \pi$	7.2
$\rho - \rho$	6.3
$\pi - \rho$	6.6
$\pi - lept$	5.6
$\rho - lept$	3.0

Table 2: Sensitivities of various channels determined without simulating final or initial state radiation and resolving the τ flight direction from final state particles

the sensitivity of the $\mathcal{O}_{opt}$ method towards $\sin^2\theta_{\rm w}$ to a sensitivity with respect to the τ polarization using the relation

$$\mathrm{P}_\tau \simeq 2(1 - 4\sin^2\theta_{\rm w})\ . \tag{3.1}$$

The calculated sensitivities to $\sin^2\theta_{\rm w}$ must be divided by 8 to get (approximately) the sensitivity for the τ polarization. This yields e.g. a sensitivity of 0.9 in the case of the $\pi - \pi$ channel, which is higher than the numbers quoted in [10] (see Appendix D). The higher sensitivity is due to the following two facts. First, our observable $\mathcal{O}_{opt}$ is optimized for a measurement of $\sin^2\theta_{\rm w}$ and therefore uses the information from the Z coupling to the τ leptons as well as the Z coupling to the electrons. Second, it uses more experimental information such as the τ direction of flight and the spin-spin correlations (as given in (2.5) and (2.6)). The observable proposed in [10], on the other hand, was optimized to measure the τ polarization using the energy spectra of single τ decay products. It is important to note that for the $\tau \to \pi\nu_\tau$ decay $x = \mathrm{E}_\pi/\mathrm{E}_{beam}$ is the optimal observable for extracting the Z coupling to τ leptons alone, but $\mathcal{O}_{opt}$ is optimized for the measurement of $\sin^2\theta_{\rm w}$. In this sense it is obvious that the calculated sensitivities of $\mathcal{O}_{opt}$ – derived from the sensitivity to $\sin^2\theta_{\rm w}$ – are larger than the sensitivities quoted in [10].

3.2 Influence of phase space cuts

Usual particle detectors do not cover the full spatial angle, at least the extreme forward and backward regions have to be excluded. Also possible background sources (e.g. Bhabha events) peak in special regions of the detector, which therefore may have to be excluded. Nonetheless not all regions of the detector have the same resolution. The sensitivity for $\pi - \pi$ events is reduced from 7.2 (see Tab. 2), corresponding to the full spatial angle, to 6.7, when the spatial angle is restricted to $|\cos\theta| < 0.7$. Therefore it would be desirable to extend the spatial volume for this analysis as far as possible to the forward and backward regions.

3.3 Production of Monte Carlo spectra with detector simulation

For the determination of the coefficients a and b in Eq. (2.7) on the detector level, event samples with full detector simulation for at least two different values of $\sin^2\theta_{\rm eff}^{\rm lept}$ are necessary. Further points can be used to verify the linear dependence (2.7). Each LEP collaboration has produced a large number of Monte Carlo events with full detector simulation for only one specific $\sin^2\theta_{\rm w}$ value close to the combined value (1.5). For getting a second or even third high statistics fully detector simulated Monte Carlo event sample with different values of $\sin^2\theta_{\rm w}$, an enormous amount of computing time would have to be provided, which is not feasible. Instead, a weighting algorithm can be used, which applies an individual weighting factor (R_i) for each of the fully detector simulated Monte Carlo events and produces distributions of $\mathcal{O}_{opt}$ for different values of $\sin^2\theta_{\rm w}$. The weighting factor is calculated from the spin-density-matrix $\mathrm{M}^{\rm Z}_{tree}$ of Eq. (A.1) as,

$$\mathrm{R}_i(\sin^2\theta'_{\rm w}) = \frac{\mathrm{M}^{\rm Z}_{tree}(\sin^2\theta'_{\rm w})}{\mathrm{M}^{\rm Z}_{tree}(\sin^2\theta_{\rm w})} \tag{3.2}$$

where $\theta_{\rm w}$ is the electroweak mixing angle with which the Monte Carlo has been produced, and $\theta'_{\rm w}$ is the value for which one desires to produce the $\mathcal{O}_{opt}$ spectrum.
By this weighting algorithm one can in principle produce a large number of different $\mathcal{O}_{opt}$ distributions, but one has to consider in the error calculation that all reweighted distributions are statistically dependent, because they are produced from the same original events.

4 Conclusions

We have presented a new approach to measure the electroweak mixing angle $\sin^2\theta_{\rm w}$ from the process $e^+e^- \to \mathrm{Z} \to \tau^+\tau^-$ with subsequent τ decays using optimal observables. The optimal observables inherently include spin correlations between the two τ leptons and exploit the available information on the τ flight direction. Our method is complementary to the well know techniques used at LEP and SLC to determine $\sin^2\theta_{\rm w}$ from various τ asymmetry measurements.

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A Spin-density-matrix

The tree-level spin-density-matrix element for the process (2.1) with Z exchange only is given below. For the full matrix element the reader is referred to [19].

$$
\begin{aligned}
\mathrm{M}^{\mathrm{Z}}_{tree} \;=\; & \frac{s(V_{\mathrm{e}}^2 + A_{\mathrm{e}}^2)}{(s - m_{\mathrm{Z}}^2)^2 + m_{\mathrm{Z}}^2\Gamma_{\mathrm{Z}}^2} \\
& \times \big(V_\tau^2\{k_0^2 + m_\tau^2 + |k|^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}) - \mathbf{S}^+\cdot\mathbf{S}^-|k|^2[1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2] \\
& + 2(\hat{\mathbf{k}}\cdot\mathbf{S}^+)(\hat{\mathbf{k}}\cdot\mathbf{S}^-)[|k|^2 + (k_0 - m_\tau)^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2] + 2k_0^2(\hat{\mathbf{p}}\cdot\mathbf{S}^+)(\hat{\mathbf{p}}\cdot\mathbf{S}^-) \\
& -2k_0(k_0 - m_\tau)\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}\,[(\hat{\mathbf{k}}\cdot\mathbf{S}^+)(\hat{\mathbf{p}}\cdot\mathbf{S}^-) + (\hat{\mathbf{k}}\cdot\mathbf{S}^-)(\hat{\mathbf{p}}\cdot\mathbf{S}^+)]\} \\
& + A_\tau^2|k|^2\{1 + (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 + \mathbf{S}^-\cdot\mathbf{S}^+[1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^2)] \\
& - 2(\hat{\mathbf{p}}\cdot\mathbf{S}^+)(\hat{\mathbf{p}}\cdot\mathbf{S}^-) + 2\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}[(\hat{\mathbf{k}}\cdot\mathbf{S}^+)(\hat{\mathbf{p}}\cdot\mathbf{S}^-) + (\hat{\mathbf{k}}\cdot\mathbf{S}^-)(\hat{\mathbf{p}}\cdot\mathbf{S}^+)]\} \\
& + A_\tau V_\tau 2|k|\{\hat{\mathbf{k}}\cdot(\mathbf{S}^+ + \mathbf{S}^-)[k_0 + (k_0 - m_\tau)(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2] + m_\tau(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}\cdot(\mathbf{S}^+ + \mathbf{S}^-)\} \\
& + cV_\tau^2 2k_0\{m_\tau\hat{\mathbf{p}}\cdot(\mathbf{S}^+ + \mathbf{S}^-) + (k_0 - m_\tau)(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\hat{\mathbf{k}}\cdot(\mathbf{S}^+ + \mathbf{S}^-)\} \\
& + cA_\tau^2 2|k|^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\hat{\mathbf{k}}\cdot(\mathbf{S}^+ + \mathbf{S}^-) \\
& + cA_\tau V_\tau 2|k|\{m_\tau[(\hat{\mathbf{k}}\cdot\mathbf{S}^+)(\hat{\mathbf{p}}\cdot\mathbf{S}^-) + (\hat{\mathbf{k}}\cdot\mathbf{S}^-)(\hat{\mathbf{p}}\cdot\mathbf{S}^+)] \\
& + 2k_0\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + 2(k_0 - m_\tau)(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\mathbf{S}^+)(\hat{\mathbf{k}}\cdot\mathbf{S}^-)\}\big)
\end{aligned}
\tag{A.1}
$$

with

$$V_f = \frac{T_3^f - 2Q^f\sin^2\theta_{\mathrm{w}}}{2\sin\theta_{\mathrm{w}}\cos\theta_{\mathrm{w}}}e\,, \tag{A.2}$$

$$A_f = \frac{T_3^f}{2\sin\theta_{\mathrm{w}}\cos\theta_{\mathrm{w}}}e\,, \tag{A.3}$$

$$c = \frac{2V_{\mathrm{e}}A_{\mathrm{e}}}{V_{\mathrm{e}}^2 + A_{\mathrm{e}}^2}\,. \tag{A.4}$$

T_3^f is the third component of the weak isospin and Q^f the charge of the fermion. The four-momenta of the τ^+ is denoted by $k = (k_0, \mathbf{k})$.

B Tau flight direction

It is well known that in the reaction

$$e^+e^- \to \tau^+(\mathbf{k})\,\tau^-(-\mathbf{k}) \to A^+(\mathrm{E}_A, \mathbf{p}_A)\,B^-(\mathrm{E}_B, \mathbf{p}_B)\,\nu_\tau\bar{\nu}_\tau \tag{B.1}$$

the direction unit vector $\hat{\mathbf{k}}$ of the τ^+ is given by

$$\hat{\mathbf{k}} \;=\; u\hat{\mathbf{p}}_A \;+\; v\hat{\mathbf{p}}_B \;\pm\; w\frac{\mathbf{p}_A \times \mathbf{p}_B}{|\mathbf{p}_A \times \mathbf{p}_B|}, \tag{B.2}$$

where hats denote unit momenta in the laboratory system. The two solutions differ only by the sign of the component w perpendicular to the plane spanned by the directions $\hat{\mathbf{p}}_A$ and $\hat{\mathbf{p}}_B$. u, v and w can be calculated as

$$u = \frac{\cos\theta_A + \hat{\mathbf{p}}_A\hat{\mathbf{p}}_B\cos\theta_B}{1 - (\hat{\mathbf{p}}_A\hat{\mathbf{p}}_B)^2} \tag{B.3}$$

$$v = -\frac{\cos\theta_B + \hat{\mathbf{p}}_A\hat{\mathbf{p}}_B\cos\theta_A}{1 - (\hat{\mathbf{p}}_A\hat{\mathbf{p}}_B)^2} \tag{B.4}$$

$$w = \sqrt{1 - u^2 - v^2 - 2uv(\hat{\mathbf{p}}_A\hat{\mathbf{p}}_B)} \quad , \tag{B.5}$$

where $\theta_{A,B}$ is the angle between the tau momenta and the momenta of the decay products A, B

$$\cos\theta_{A,B} = \frac{2k_0\mathrm{E}_{A,B} - m_{A,B}^2 - m_\tau^2}{2|p_{A,B}||k|} \quad . \tag{B.6}$$

Three cases must be considered in the reconstruction. When both τ leptons decay to charged hadrons only, it is possible to resolve the ambiguity of Eq. (B.2) as proposed in [24]. If at least one τ decays into a hadronic final state which contains one or more neutral hadrons, the ambiguity cannot be resolved. If one τ lepton decays leptonically, eq. (B.2) is only an approximation of the τ flight direction.

However, the magnitude of w, the component perpendicular to the plane spanned by p_A and p_B, is of the order m_τ/m_Z. The sign of the ambiguity can be experimentally resolved only in hadronic τ decays when all final state particles (apart of the ν_τ) are charged. For the τ decay channels considered only $\tau \to \pi\nu_\tau$ decays on both sides applies to such a treatment. However, perfectly resolving this ambiguity in $\tau \to \pi\nu_\tau$ decays, leads to a small increase of the sensitivity to $\sin^2\theta_\mathrm{w}$ of 4%. Therefore one could average over both possible τ flight directions which is equivalent to setting $w \equiv 0$.

C Spin vectors

The knowledge of the τ flight directions allows to reconstruct the τ rest system. It is then possible to calculate the most probable direction, called $\mathbf{S}^\pm$, of the τ spin vectors in the τ rest system. These spin operators, first calculated in [20], are given below. The four-momenta of the $\tau^\pm$ are denoted by $k^\pm = (k_0, \pm\mathbf{k})$.

a) $\tau \to \ell\nu_\ell\nu_\tau$

$$\mathbf{S}^\pm_{\tau\to\ell\nu_\ell\nu} = \frac{4c_\pm - m_\tau^2}{3m_\tau^2 c_\pm - 4c_\pm^2}\left(\pm m_\tau \mathbf{p}_{\ell^\pm} - \frac{c_\pm + E_{\ell^\pm}m_\tau}{k_0 + m_\tau}\mathbf{k}\right) \tag{C.1}$$

$$c_\pm = k_0 E_{\ell^\pm} \mp \mathbf{k}\cdot\mathbf{p}_{\ell^\pm} \tag{C.2}$$

Here $p_{\ell^\pm} = (E_{\ell^\pm}, \mathbf{p}_{\ell^\pm})$ is the four-momentum of the outgoing lepton and lepton masses have been neglected.

b) $\tau \to \pi\nu_\tau$

$$\mathbf{S}^{\pm}_{\tau\to\pi\nu} = \frac{2}{m_\tau^2 - m_\pi^2}\left(\mp m_\tau \mathbf{p}_{\pi^\pm} + \frac{m_\tau^2 + m_\pi^2 + 2m_\tau E_{\pi^\pm}}{2(E_\tau + m_\tau)}\mathbf{k}\right). \tag{C.3}$$

Here $p_{\pi^\pm} = (E_{\pi^\pm}, \mathbf{p}_{\pi^\pm})$ denotes the pion's four-momentum.

c) $\tau \to \rho\nu_\tau \to \pi\pi^0\nu_\tau$

$$\mathbf{S}^{\pm}_{\tau\to\pi\pi^0\nu} = \mp\frac{1}{(k_\pm H_\pm) - m_\tau^2(p_{\pi^\pm} - p_{\pi^0})^2} \mp H_0^\pm \mathbf{k} + m_\tau \mathbf{H}^\pm + \frac{\mathbf{k}(\mathbf{k}\cdot\mathbf{H}^\pm)}{(E_\tau + m_\tau)} \tag{C.4}$$

$$(H^\pm)^\nu = 2(p_{\pi^\pm} - p_{\pi^0})^\nu (p_{\pi^\pm} - p_{\pi^0})^\mu (k_\pm)_\mu + (p_{\pi^\pm} + p_{\pi^0})^\nu (p_{\pi^\pm} - p_{\pi^0})^2 . \tag{C.5}$$

Here $\mathrm{p}_{\pi^\pm}$ *and* p_{π^0} are the four-momenta of charged and neutral pion.

D Sensitivity of the optimal observable $\boldsymbol{\omega}$ for the determination of the τ polarization

We show here the calculation of the sensitivity of the optimal observable ω for $\tau \to \pi\nu_\tau$ and leptonic τ decays, because the explicit form of the optimal observables are not given in [10].

The sensitivity of the optimal observable ω is calculated as

$$\begin{aligned} S^2 &= \frac{1}{N\sigma^2} \\ &= \langle \frac{\omega^2}{(1 + p_0 \cdot \omega)^2} \rangle \\ &= \int \frac{\omega^2}{(1 + p_0 \cdot \omega)^2} (f(\xi) + p_0 \cdot g(\xi))\, \mathrm{d}\xi \\ &= \int \frac{g^2(\xi)}{f(\xi) + p_0 \cdot g(\xi)}\, \mathrm{d}\xi \,, \end{aligned} \tag{D.1}$$

where ξ denotes the set of observables which contains all information on $\sin^2\theta_\mathrm{w}$.
For the π channel with

$$W(x) = 1 + P_\tau(2x - 1)\,, \quad x = \mathrm{E}_\pi/\mathrm{E}_{beam} \tag{D.2}$$

the sensitivity is calculated as follows. Setting $p_0 = 0$, which is a good approximation for small values of the polarization, the sensitivity for the π channel ($g(x)_\pi = 2x - 1$) is

$$\int_0^1 g^2(x)_\pi \,\mathrm{d}x = \left[\frac{4}{3}x^3 - 2x^2 + x\right]_0^1 = \frac{1}{3}\,, \tag{D.3}$$

resulting in a sensitivity

$$S_\pi = \frac{1}{\sqrt{3}} = 0.58 \ . \tag{D.4}$$

In the case of leptonic τ decays the dependence on P_τ is

$$W(x) = \frac{1}{3}\left((5 - 9x^2 + 4x^3) + P_\tau(1 - 9x^2 + 8x^3)\right) \tag{D.5}$$

resulting in a sensitivity of

$$S_{lept} = 0.22 \ . \tag{D.6}$$

These numerical results agree with the numbers quoted in [10] (see Table 3). For the ρ and a_1 channels the sensitivities calculated in [10] are also shown in Tab. 3.

Channel	Sensitivity	Sensitivity (with $\vec{\tau}$)
π	0.58	0.58
ρ	0.49	0.58
a_1	0.45	0.58
$lept$	0.22	0.27

Table 3: Sensitivities achieved applying optimal observables without and with the use of the τ flight direction [10]

It is notable that these sensitivities are calculated for single τ decays only, but at LEP the τ leptons are produced in pairs. As soon as both τ decays could be identified – which is likely due to the high detection efficiencies presently achieved by the LEP experiments – one has to take into account that the two τ helicities are opposite, neglecting m_τ^2/m_Z^2 effects, and consequently the polarization measurements performed on both τ's are correlated. In [10] it is shown, that in the case of two identified τ leptons the optimal observables, calculated for single identified τ leptons, can be modified to remain optimal. The quoted value for the $\pi - \pi$ channel, with correct treatment of the spin correlations, therefore reduces to a sensitivity of 0.73 compared to the value 0.81 ($\sqrt{2} \cdot 0.58 = 0.81$) when each τ decay is treated individually and correlations are neglected.

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